Activities for Trigonometric

WORK #1

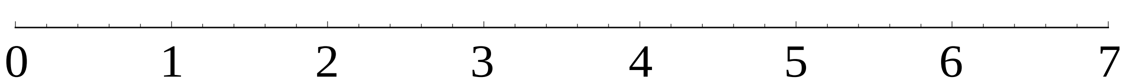
- **Basic knowledge of circle, numbers in radii, measuring angles in units of radii (radians).**

1. Basic knowledge of circle:
   a. Use a compass (or the lid of a can) to draw a circle as carefully as you can. Use a ribbon, measuring tape, wikistick, or similar material (that you can fold around the circle), to prepare a measuring tape that is in units of circle radii and has a length of at least 4 radii. Subdivide and mark each length of a radius into ten equal parts as carefully as possible.
   b. Use the measuring tape you prepared to measure as carefully as you can the number of radii in the half circle circumference. What is the relationship between this number and the value of pi given to you by the calculator?
   c. Will you get the same result if you use another circle (with a ribbon in units of the radius of the other circle)? (You may compare the result of your group with that of other groups.)
   d. According to the measurement you got in the previous part, approximately how many radii are there in the circumference of your circle?
   e. Explain the relationship between what you did in the previous parts and the well-known formula $C = 2\pi r$.

2. In this exercise you can use the fact that π is approximately 3.14. You should be able to do the exercise without using a calculator.
   a. Explain why $\pi/3$ is a number very close to 1.
   b. Will $\pi/3$ be greater or less than 1? Explain.

3. Locate as carefully as possible each of the following numbers on the real line provided below (use the fact that π is slightly larger than 3; $\pi \cong 3.14 > 3$; You should be able to do the exercise without using a calculator).

a. $\pi/3$	b. $2\pi/3$	c. π	d. $4\pi/3$
e. $5\pi/3$	f. 2π	g. $\pi/4$	h. $\pi/2$

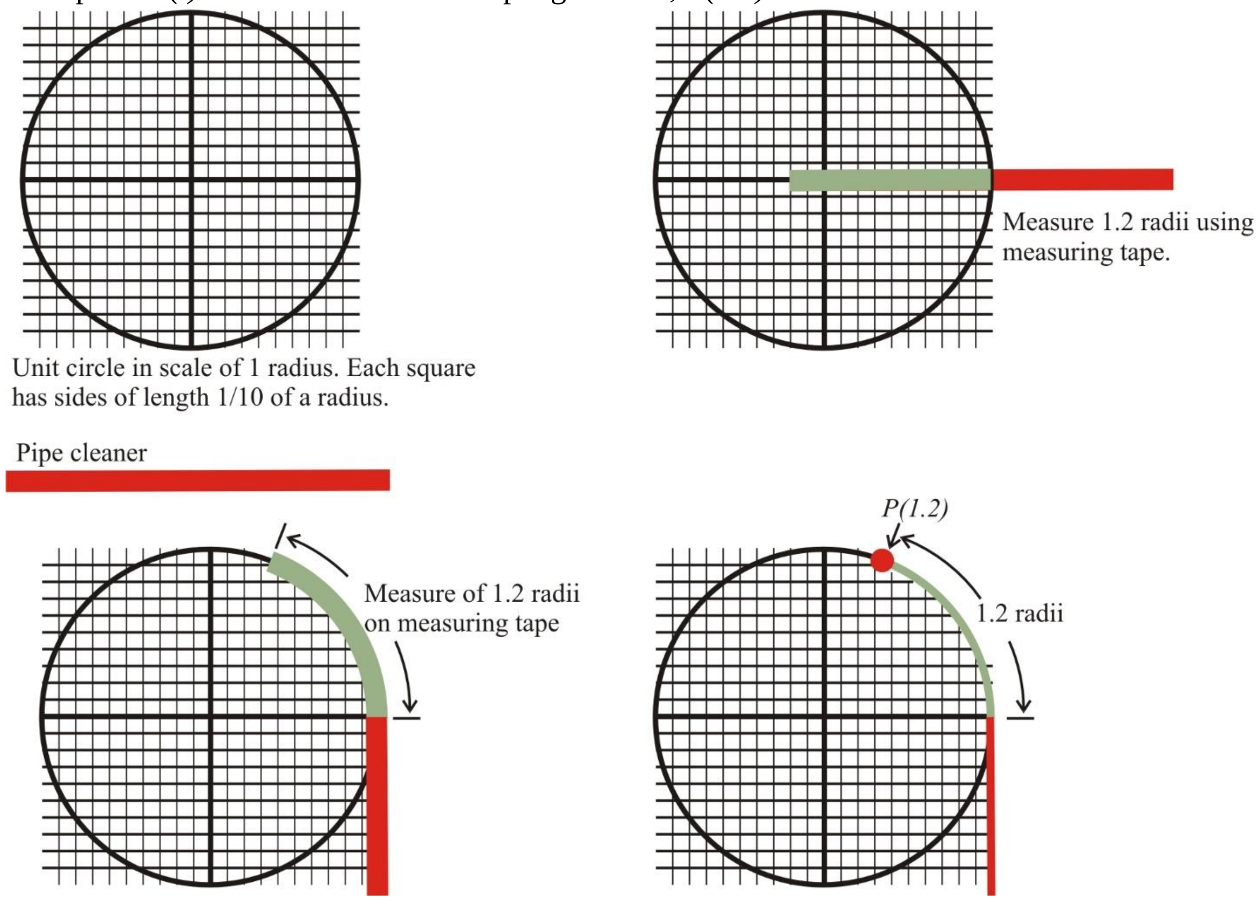


4. Locate as carefully as possible each of the following numbers on the real line provided below (use the fact that π is slightly larger than 3; $\pi \cong 3.14 > 3$):

a. 1	b. 2	c. 3	d. $\pi/2$
e. 4	f. 5	g. 6	h. $3\pi/2$

$\frac{\pi}{3}$ $\frac{2\pi}{3}$ π $\frac{4\pi}{3}$ $\frac{5\pi}{3}$ 2π

5. Point determined by a number: Given a circle and any real number *t*, that number determines a point *P(t)* on the circle. For example given 1.2, *P(1.2)* is determined as follows:

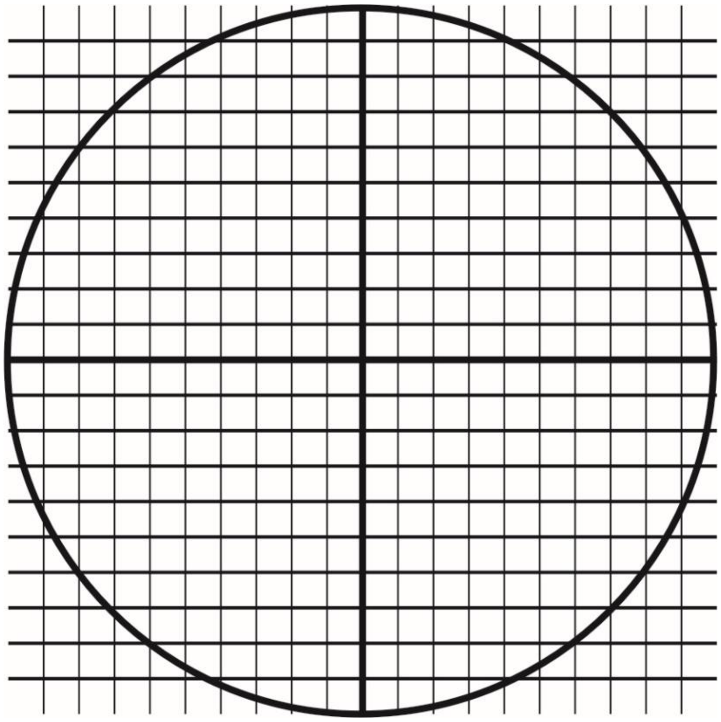


In general, you put the circle on a Cartesian plane and use the radius of the circle as a unit of measurement. You start at the rightmost point on the circle; this is the point (1,0) (coordinates in units of radii) and measure an arc of length *t*. If $t \geq 0$ the arc is measured counter-clockwise and if $t < 0$ the arc is measured clockwise. The point where you end is *P(t).*

In each of the following problems, use the circle in the figure below and a piece of thread or wikistick to locate, as carefully as you can, the *point P(t)* on the circle determined by the given number *t* (in units of radii). (Tip: Note that each square has sides that measure 1/10 of a radius.)

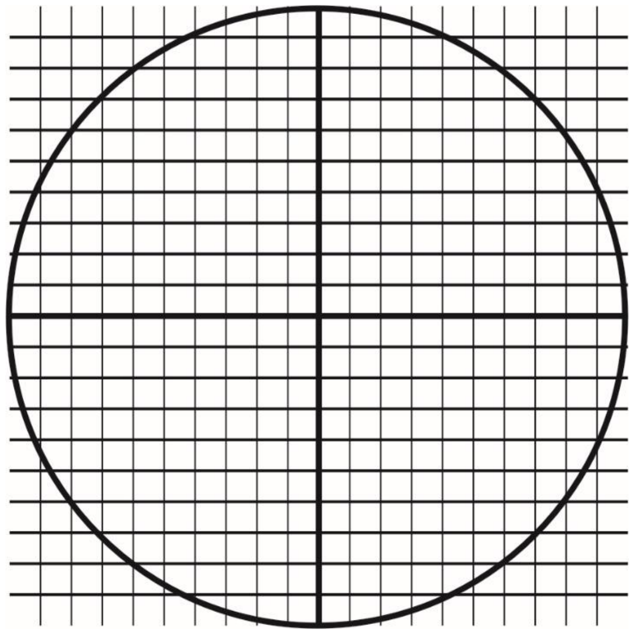
a. $\frac{1}{2}$ radii b. 2 radii c. -2 radii d. $-\frac{3}{4}$ radii

e. 0.4 radii f. $\pi - 0.4$ radii g. $\pi + 0.4$ radii h. 1.6 radii



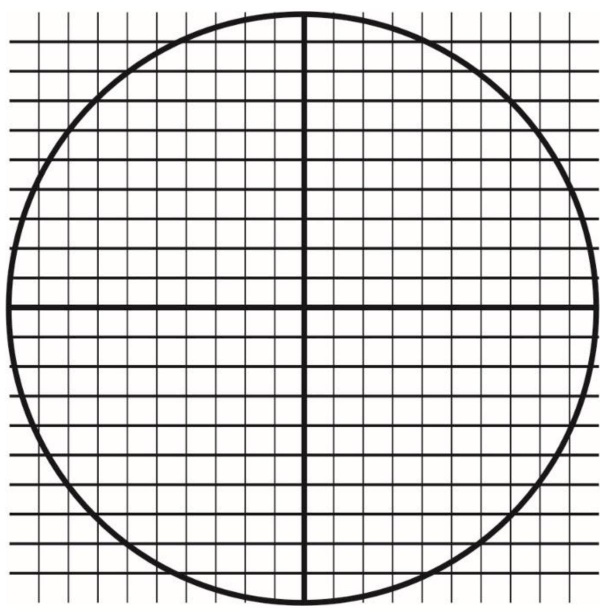
6. In each of the following problems, use the circle in the figure below and a piece of thread or wikistick to locate, as carefully as you can, the *point P(t)* on the circle determined by the given number *t*. The radius of the circle is used as a unit of measurement. (Tip: Note that each square has sides that measure 1/10 of a radius.)

a. 0	b. 1	c. 2	d. 3
e. 4	f. 5	g. 6	h. –6

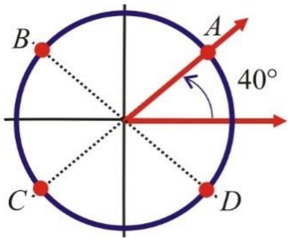


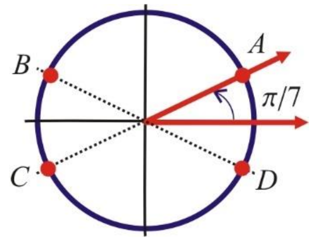
7. When a circular arc length is measured in units of radii, it is also said to be measured in *radians*. For each value of t that is given, locate in the circle given below the point $P(t)$ that corresponds to it. (Remember that a semicircle measures π times the radius.)

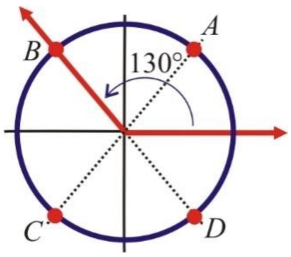
   a. 1 radian
   b. $1+\pi$ radians
   c. $1-\pi$ radians
   d. $1+2\pi$ radians
   e. $\pi-1$ radians
   f. $2\pi-1$ radians
   g. -1 radians
   h. $-1+\pi$ radians

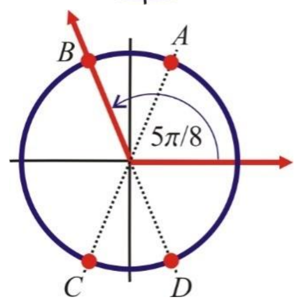


8. A point on a circle determines three other points on the circle, reflecting through each axis and through the origin. In each of the following cases indicate an angle (in standard position) that determines each of the points *A*, *B*, *C*, *D*.

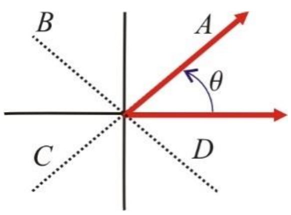
a. 

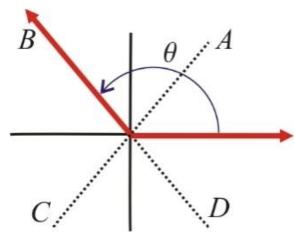
b. 

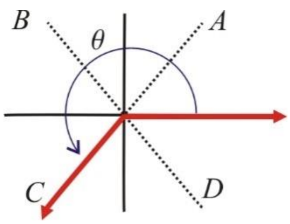
c. 

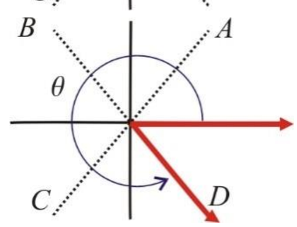
d. 

9. In each of the following cases an angle is given and its measurement θ that we assume is in radians. Indicate the measurement in radians of an angle that is coterminal with the side marked *A*, *B*, *C*, *D* (the answer is going to be an expression in terms of π).

a. 

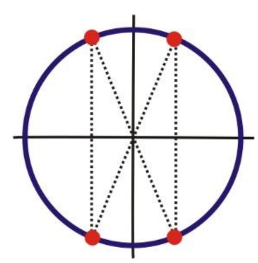
b. 

c. 

d. 

10. For any value of t explain the following relation in terms of reflections across the axes and/or rotations of $180°$. You can use the figure below to explain to yourself that the relationship is the same no matter what quadrant $P(t)$ is in (simply locate $P(t)$ at one of the four locations, then find the location of the other).

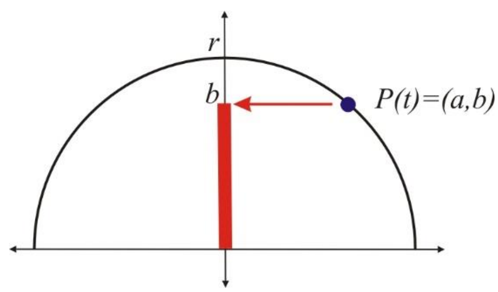
   a. What is the relation between $P(t)$ and $P(-t)$?
   b. What is the relation between $P(t)$ and $P(\pi-t)$?
   c. What is the relation between $P(t)$ and $P(2\pi-t)$?
   d. What is the relation between $P(t)$ and $P(t+\pi)$?



WORK #2

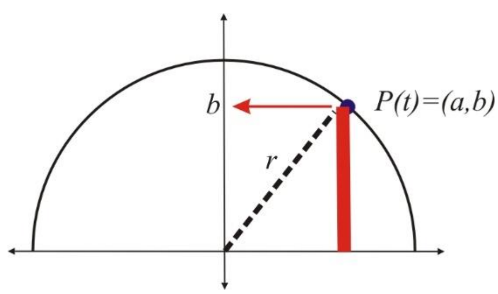
- **The definition of sine using unit circle.**
- **The definition of cosine using unit circle.**

1. Let $P(t)=(a,b)$ be a point in a circle with center at the origin (assume that *a* and *b* are measured in some unit [cm., inches, …] which need not be radii). The projection of the point $P(t)=(a,b)$ on the *y*-axis **(in units of radii)** is the fraction that *b* makes of a radius (*b*/*r*). Its graphical representation is (note that the graphical representation includes an arrow and a darkened vertical segment):



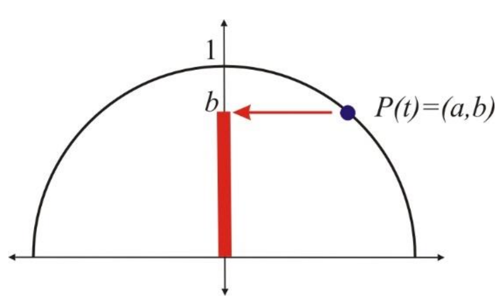
$$\text{fraction of radius} = \frac{b}{\text{radius}} = \frac{b}{r}$$

It can also be represented in triangular shape as follows:



$$\text{fraction of radius} = \frac{b}{\text{radius}} = \frac{b}{r}$$

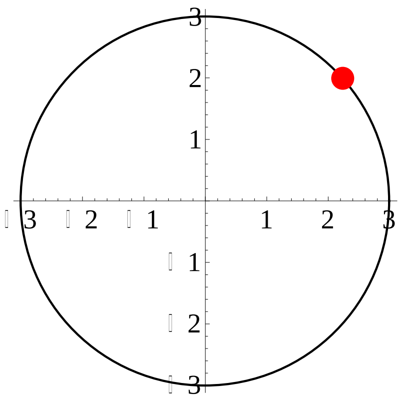
Note that if the coordinate system is in radii units, then the circle is of radius 1 unit (**unit circle**) and the projection of $P(t)=(a,b)$ on the *y*-axis is *b*.



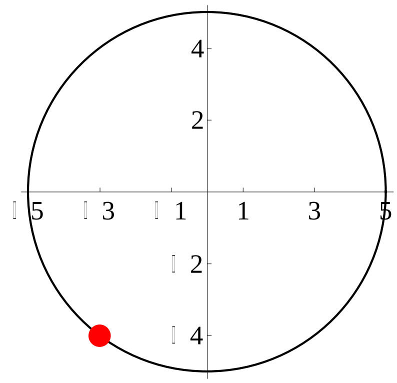
$$\text{fraction of radius} = \frac{b}{\text{radius}} = \frac{b}{1} = b$$

In each of the following cases represent graphically (add arrow and darkened vertical segment) and find the projection (a number) of *P(t)* on the *y* axis in units of radii.

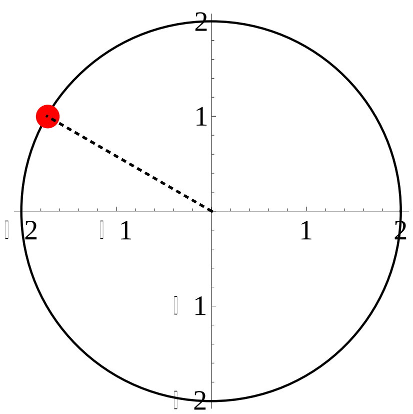
a.



b.



c.



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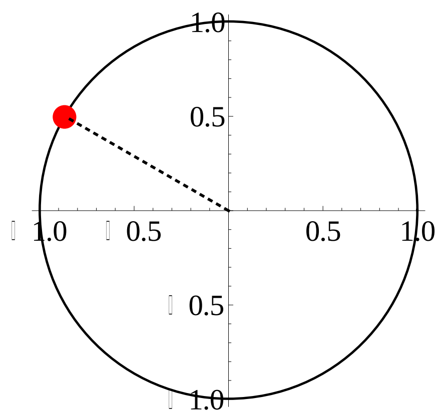
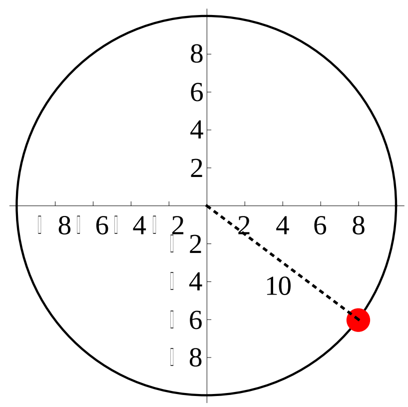
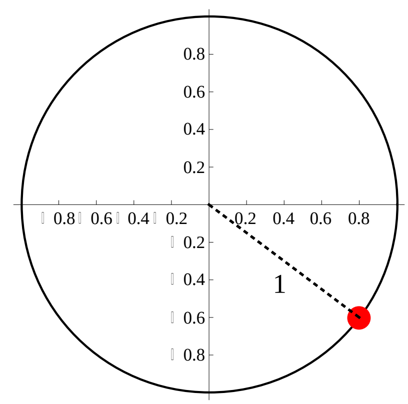


e.

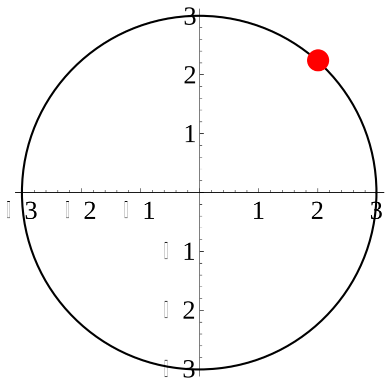


f.

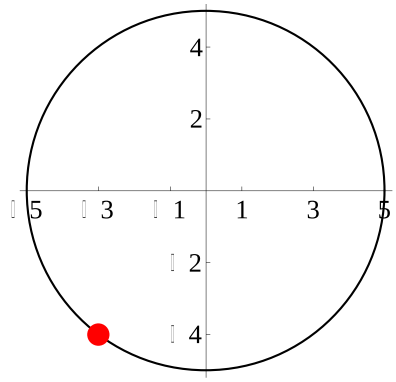


2. If the point $P(t) = (a, b)$ is in a circle and the radius of the circle is used as a unit of measurement then the projection **in radii units** of the point $P(t) = (a, b)$ **on the *x-axis*** is defined and represented graphically analogous to what was done in the previous problem. That is, in this case the projection on the axis of x is going to be a / r. In each of the following problems use the radius of the circle as a unit of measurement to find and graphically represent the projection of the given point on the *x-axis*.

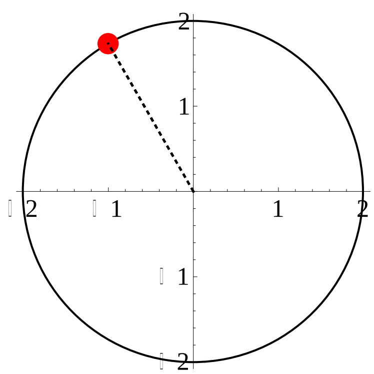
a.



b.



c.



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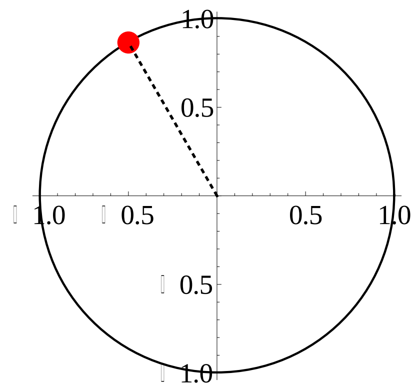
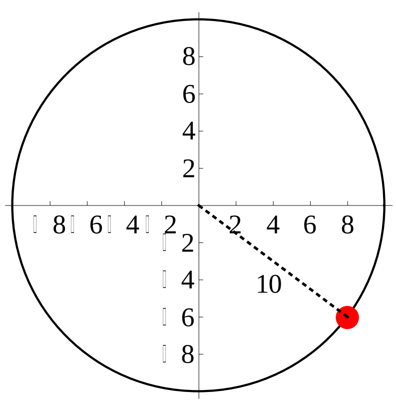
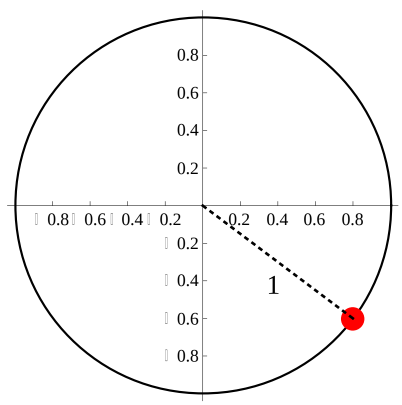


e.



f.

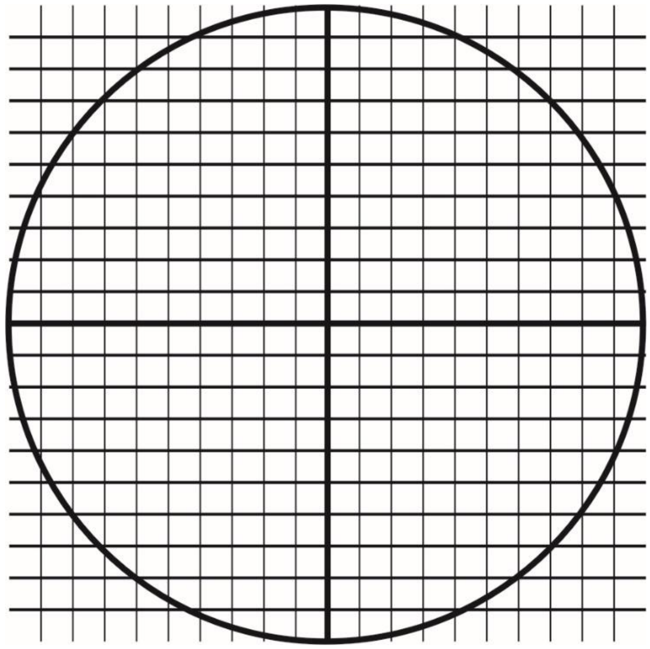


3. Given any real number *t*, the sine of *t* is defined, $\sin(t)$, as follows:
   - First you locate *P(t)* in a circle
   - Then you project *P(t)* on the *y-axis* and give the result in units of radii

   For each of the following problems use the circle in the figure below and a piece of thread or wikistick to approximate each of the following sine values as closely as you can:

   a. $\sin\left(\frac{1}{2}\right)$ b. $\sin(2)$ c. $\sin(-2)$ d. $\sin\left(-\frac{3}{4}\right)$

   e. $\sin(0.4)$ f. $\sin(\pi-0.4)$ g. $\sin(\pi+0.4)$ h. $\sin(1.6)$



4. Remembering that a semicircle has a length of π times the radius, you must be able to find the following sine values without using a calculator or having to memorize them. Imagine locating the corresponding point *P(t)* and then project on the *y*-axis.

   a. $\sin(0)$ b. $\sin(\pi)$ c. $\sin(\pi/2)$ d. $\sin\left(-\dfrac{3\pi}{2}\right)$

   e. $\sin(-\pi/2)$ f. $\sin(2\pi)$ g. $\sin(5\pi/2)$ h. $\sin(-3\pi)$

5. You should be able to do this exercise without using a calculator, thread, or wikistick. Use only your imagination (imagine locating and then projecting) and what you know about π. Put in order from lowest to highest: $\sin(1)$, $\sin(1/10)$, $\sin(3)$, $\sin(-4)$, $\sin(4)$, $\sin(6)$

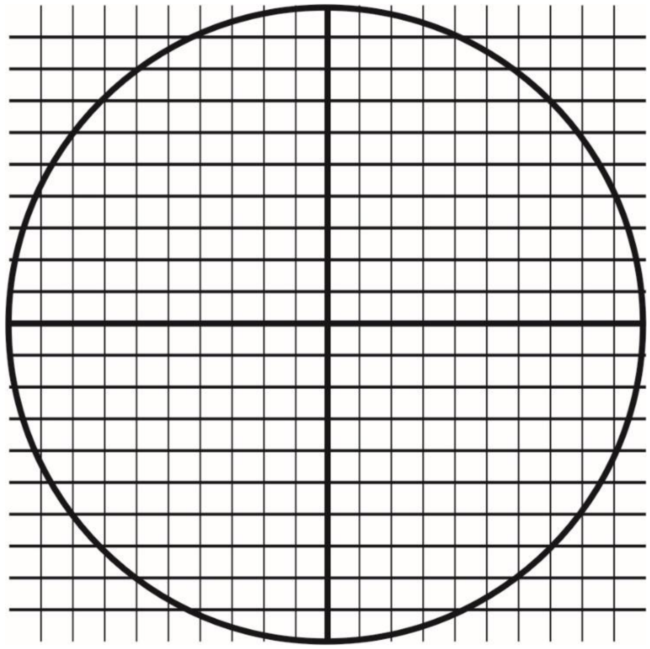
6. Given any real number *t*, the cosine of *t* is defined, $\cos(t)$ as follows:

   - First you locate *P(t)* in a circle
   - Then you project *P(t)* on the *x* axis and give the result is given in units of radii

   For each of the following problems use the circle in the figure below and a piece of thread or wikistick to approximate each of the following cosine values as closely as you can:

   a. $\cos\left(\dfrac{1}{2}\right)$ b. $\cos(2)$ c. $\cos(-2)$ d. $\cos\left(-\dfrac{3}{4}\right)$

   e. $\cos(0.4)$ f. $\cos(\pi-0.4)$ g. $\cos(\pi+0.4)$ h. $\cos(1.6)$



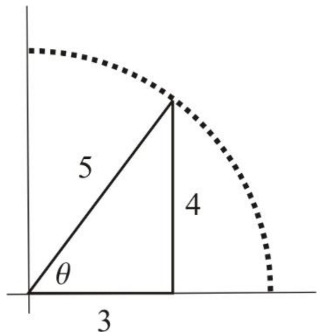
7. Remembering that a semicircle has a length of π times the radius, you must be able to find the following cosine values without using a calculator or having to memorize them. Imagine locating the corresponding point *P(t)* and then project on the *x-axis*.

   a. $\cos(0)$ b. $\cos(\pi)$ c. $\cos(\pi/2)$ d. $\cos\left(-\frac{3\pi}{2}\right)$

   e. $\cos(-\pi/2)$ f. $\cos(2\pi)$ g. $\cos(5\pi/2)$ h. $\cos(-3\pi)$

8. You should be able to do this exercise without using a calculator, thread, or wikistick. Use only your imagination (imagine locating and then projecting) and what you know about π. Put in order from lowest to highest: $\cos(1)$, $\cos(1/10)$, $\cos(3)$, $\cos(-4)$, $\cos(4)$, $\cos(6)$

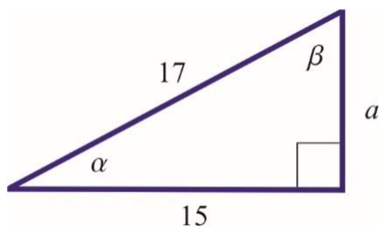
9. The definition of sine and cosine can be extended directly to the acute angles of a right triangle if we think of hypotenuse as the radius of a circle (see figure). An example is shown below. Consider an **acute angle** θ of a right triangle with opposite side of magnitude 4, adjacent side of magnitude 3, and hypotenuse of magnitude 5. Imagine the angle of interest θ in standard position and a circle generated by the hypotenuse of the triangle, as suggested by the following figure. Note that we use the same symbol, θ, to denote the angle and its measurement.



So, projecting as always,

$$\sin(\theta) = \frac{\text{projection over the } y \text{ axis}}{\text{radius}} = \frac{\text{opposite}}{\text{radius}} = \frac{4}{5}, \qquad \cos(\theta) = \frac{\text{projection over the } x \text{ axis}}{\text{radius}} = \frac{\text{adjacent}}{\text{radius}} = \frac{3}{5}$$

Find the sine and cosine of the angles α and β in the figure (Remember the Pythagorean Theorem).

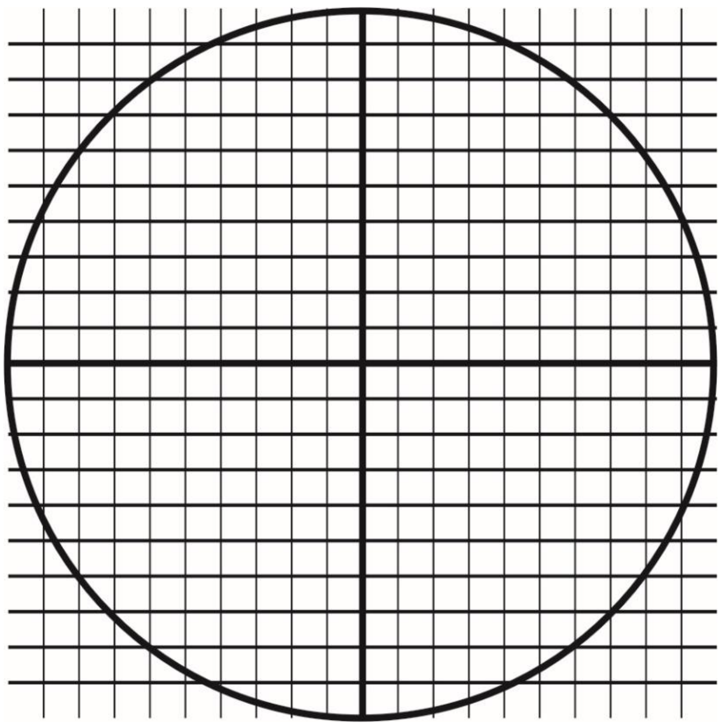


WORK #3

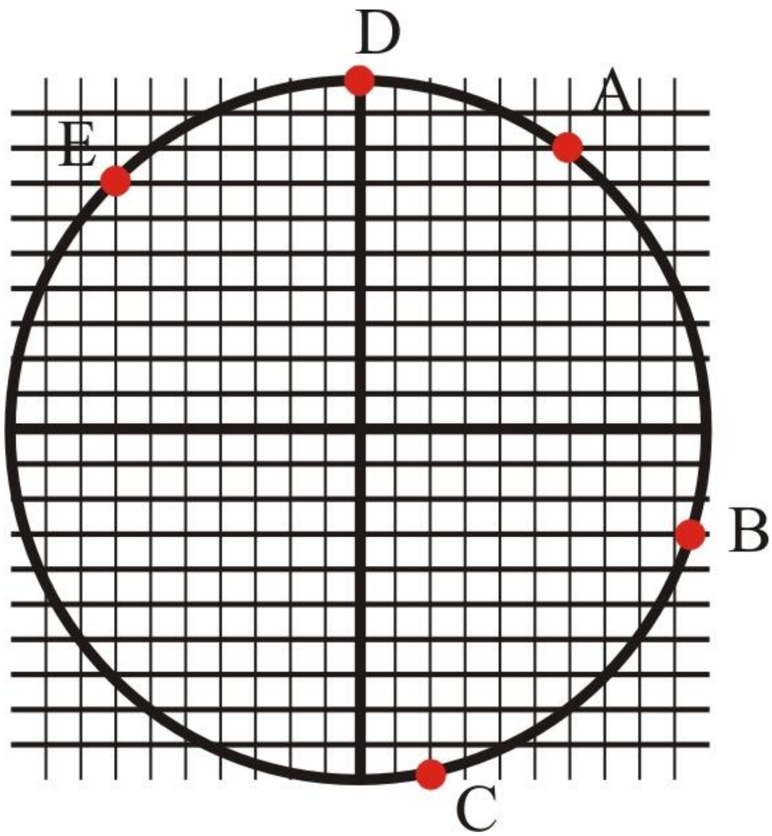
- **Making inverse sine and inverse cosine using unit circle.**

The sine and cosine functions take an angle as input and produce as output a number between 1 and -1. This group work seeks to reverse this process. Problem 1 starts with a number between 1 and -1 as input and points are produced in a unit circle as output (from number to point). Problem 2 starts with a point on a unit circle as input and produces an angle as output (point to angle). In problem 3, problems 1 and 2 come together to finally reverse sine and cosine: start with a number and end with an angle (from number to angle).

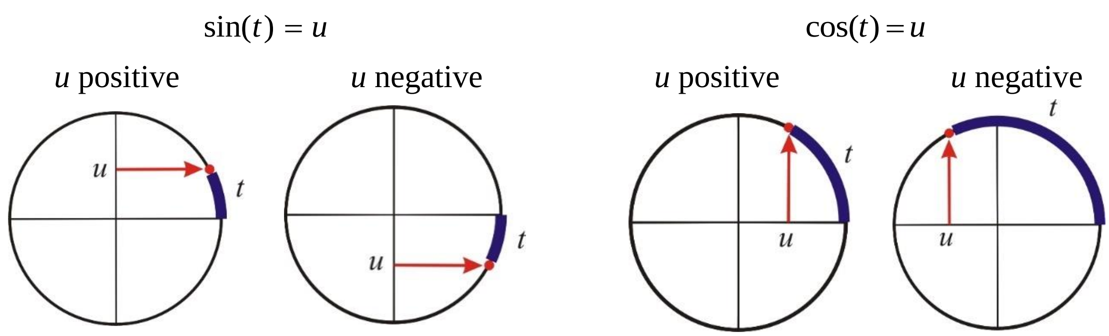
1. (From number to point) in each of the following problems use the circle below to darken all the dots $P(t)=(x,y)$ in the circle that fulfill the given condition. Label the darkened dots.
   a. $y=0.6$
   b. $\sin(t)=-0.3$
   c. $x=-0.4$
   d. $\cos(t)=0.7$
   e. $\sin(t)=0$
   f. $\cos(t)=1$
   g. $\sin(t)=-1$
   h. $\cos(t)=1.4$



2. (Point to angle) For each of the points *P(t)* in the given circle, use a piece of thread (or a wikistick) to approximate as closely as you can a value of *t* (positive, negative, or zero) that corresponds to the point. Remember that *t* represents the measurement in radians of an angle in standard position.



3. The sine and cosine functions take an angle t as input and produce as output a number u between one and minus one. In this and the next problem, that process is reversed. A number u between one and minus one will be taken as input and an angle t will be produced as an output as the following figure suggests.



Use a piece of thread (or wikistick) and the circle below, and follow the above procedure to approximate as closely as you can a value of t where:

a. $\sin(t) = -0.4$

b. $\sin(t) = 0.6$

c. $\cos(t) = -0.3$.

d. $\cos(t) = 0.7$

4. Conditions for uniqueness when reversing sine:

   a. Use a drawing of a circle to explain why each of the problems $\sin(t) = -0.4$, $\sin(t) = 0.6$ has two different solutions between 0 and 2π, and find such solutions. Explain why in each case there are infinitely many possible values of *t* that satisfy the given condition.

   b. Use the smallest possible circular arc (as the figure above suggests) to find answers to problems 3a and 3b that are numbers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

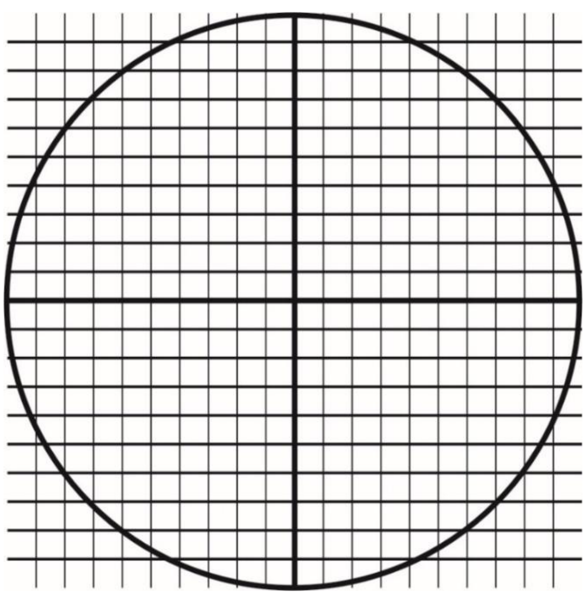
   c. Explain why if you insist that the answer be a number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ then problems 3a and 3b have only one answer.

   d. The unique answers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ of problems 3a and 3b are called $\sin^{-1}(-0.4)$ ("inverse sine of -0.4") and $\sin^{-1}(0.6)$ ("inverse sine of 0.6") respectively. In general, for a given number *u* one defines $\sin^{-1}(u)$ in a similar fashion. Use a piece of thread or wikistick to approximate as best you can $\sin^{-1}(-0.7)$.

   e. Use a thread or wikistick to approximate as best you can $\sin^{-1}(0.3)$.

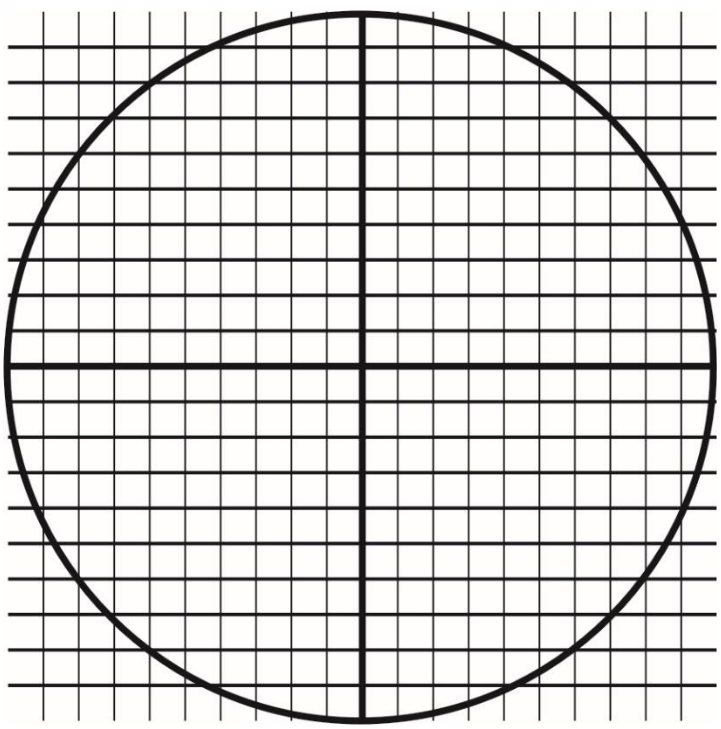
   f. Inverse sine is a function because each possible input is assigned a unique output. What is the domain (possible inputs) and range (possible outputs) of the inverse sine function?

   g. Explain why in general, if *y* is a number between 1 and -1 then $\sin^{-1}(y)$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is *y*.



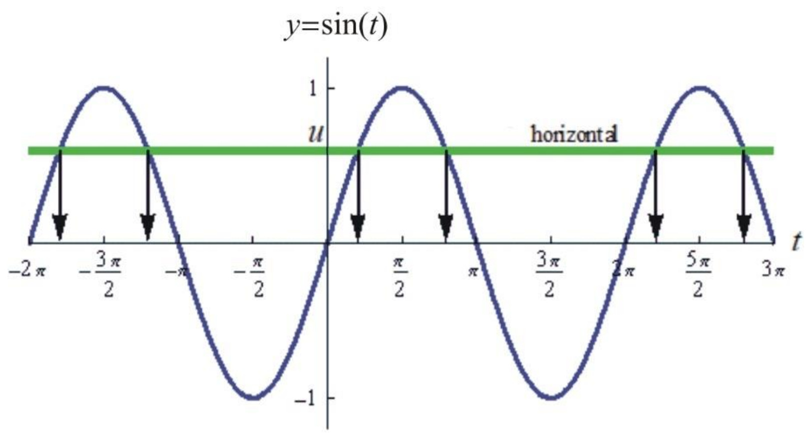
5. Inverse cosine:

   a. Use a drawing of a circle to explain why each of the problems $\cos(t) = -0.3$, $\cos(t) = 0.7$ has two different solutions between 0 and 2π, and find such solutions. Explain why in each case there are infinitely many possible values of t that satisfy the given condition.
   b. Use the smallest possible circular arc (as the figure in problem 3 suggests) to find answers to problems 3c and 3d that are numbers between 0 and π.
   c. Explain why if you insist that the answer must be a number between 0 and π then problems 3c and 3d have only one answer.
   d. The unique answers between 0 and π are called $\cos^{-1}(-0.3)$ ("inverse cosine of -0.3") and $\cos^{-1}(0.7)$ ("inverse cosine of 0.7") respectively. In general, for a given number *u*, one defines $\cos^{-1}(u)$ in a similar way. Use a thread or wikistick to approximate as best as you can $\cos^{-1}(-0.4)$.
   e. Use a thread or wikistick to approximate as best as you can $\cos^{-1}(0.6)$.
   f. Inverse cosine is a function because each possible input is assigned a unique output. What is the domain (possible inputs) and range (possible outputs) of the inverse cosine function?
   g. Explain why in general, if x is a number between 1 and -1 then $\cos^{-1}(x)$ is **the** angle between 0 and π whose cosine is x.



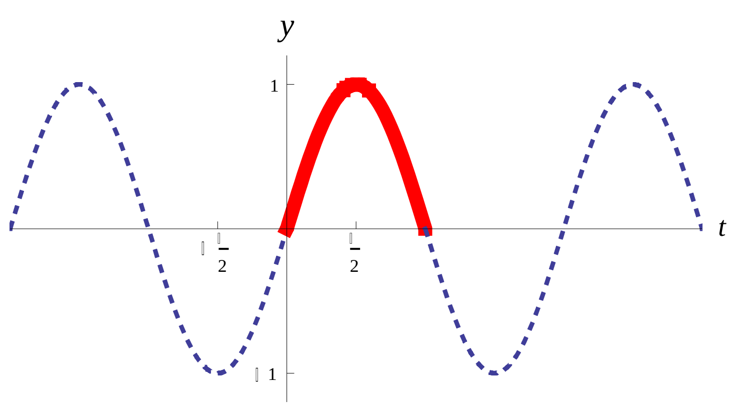
In the following problems, the inverse trigonometric functions, inverse sine and inverse cosine are studied, from their graph.

6. Inverse sine function using graphs:
   a. Use the figure below to explain why the function $f(t) = \sin(t)$ it has no inverse function. (Hint: A function is invertible when each output of the function corresponds to only one input.)
   b. Use the unit circle to explain why the sine function has no inverse function.

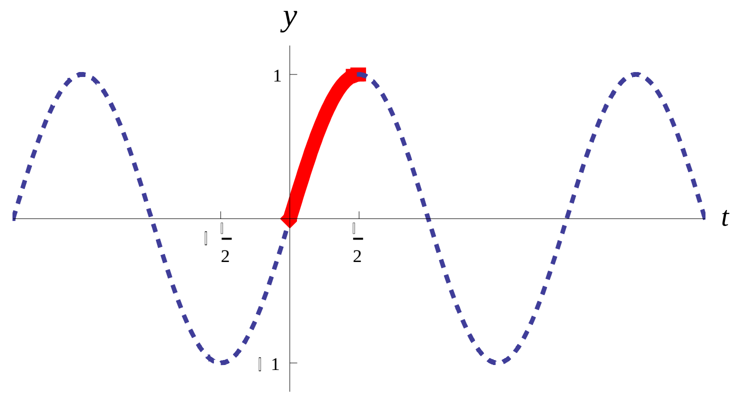


7. Explain why each of the graphs below (only the thicker portion of the sine graph that is represented) does NOT allow each output of the sine function to be assigned exactly one corresponding input. Also explain why each of the functions in parts a and b below is different from the sine function.

a.



b.



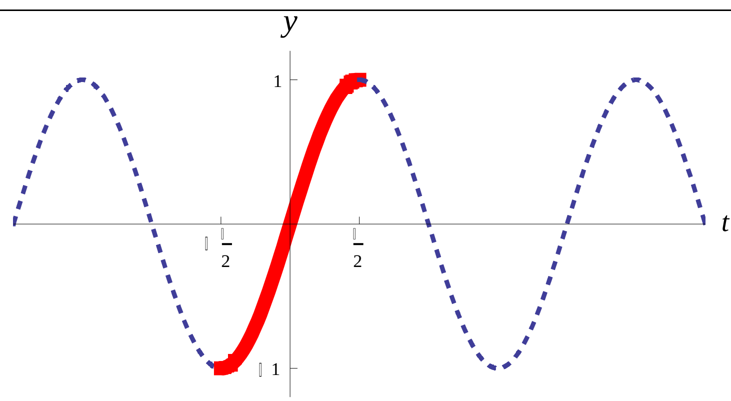
8. Interpret each part of the above problem using the unit circle.

9. Restricting the sine function:

   a. Explain why the graph below (only the thickest portion of the represented sine graph) allows each output of the sine function to be assigned exactly one corresponding input. The inverse function of that **restricted sine function** is called the inverse sine function.

   b. Interpret the above in terms of the unitary circle.

   c. Use the given portion of the graph to draw the graph of the inverse sine: $y = \sin^{-1}(t)$.

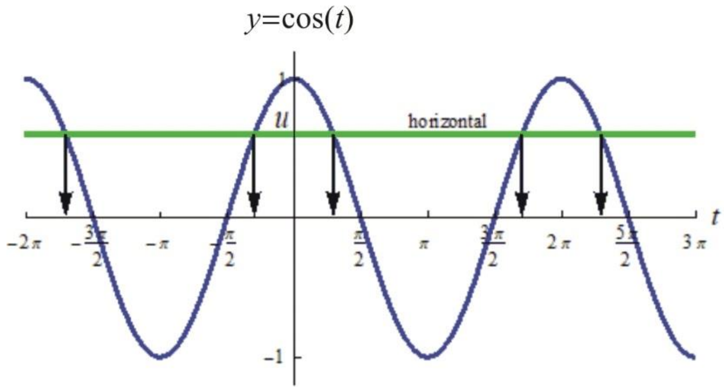


   d. What is the domain (possible inputs) and range (possible outputs) of the inverse sine function?

10. Inverse cosine function using graphs:

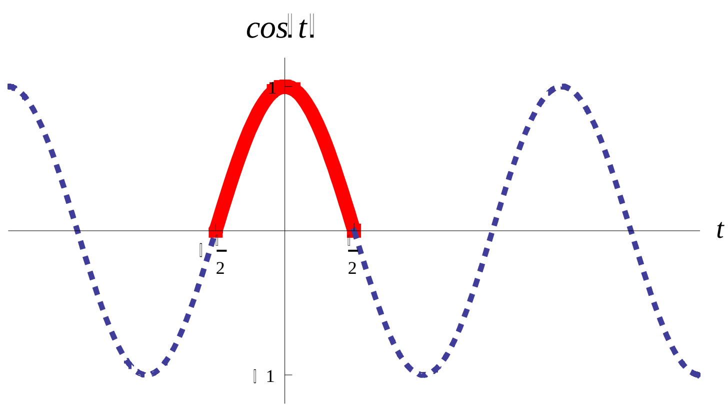
   a. Use the figure below to explain why the function $f(t) = \cos(t)$ has no inverse function. (Hint: A function is invertible when each output of the function corresponds to only one input.)

   b. Use the unit circle to explain why the cosine function has no inverse function.

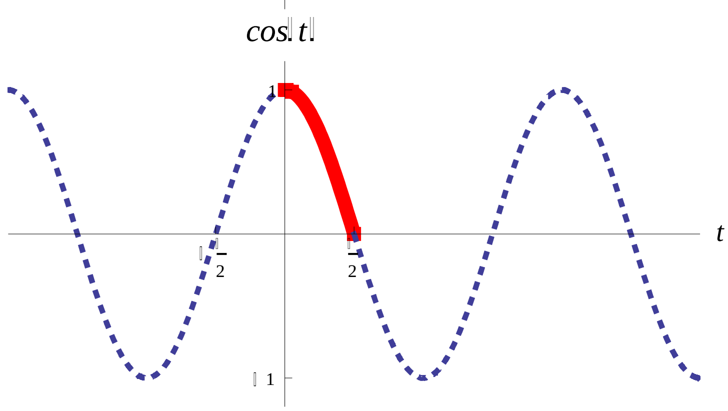


11. Explain why each of the graphs below (only the thicker portion of the represented cosine graph) does NOT allow each output of the cosine function to be assigned exactly a corresponding input.

a.



b.



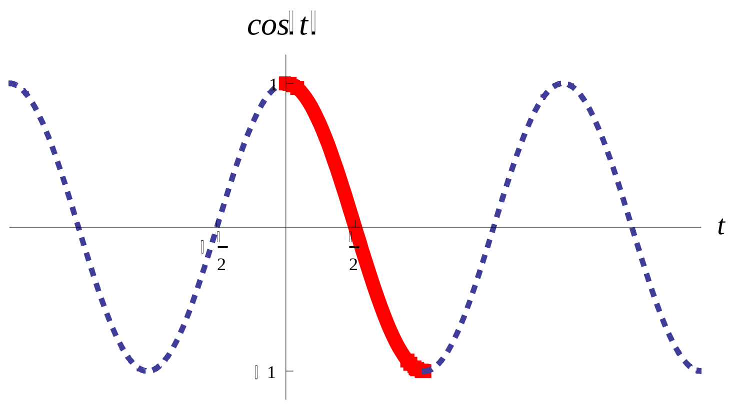
12. Interpret each of the parts of the above problem using the unit circle.

13. Restricting the cosine function:
    a. Explain why the graph below (only the thicker portion of the represented cosine graph) allows each output of the cosine function to be assigned exactly one corresponding input.

The inverse function of that **restricted cosine function** is called the inverse cosine function.

b. Interpret the above in terms of the unit circle.

c. Use the given portion of graph to draw the inverse cosine graph: $y = \cos^{-1}(t)$



d. What is the domain (possible inputs) and range (possible outputs) of the inverse cosine function?