Using APOS theory to improve students' conceptual understanding of mathematical concepts: the case of trigonometric functions in secondary schools

Vahid Borji

Faculty of Mathematics and Physics, Department of Mathematics Education

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APOS Theory

- The acronym APOS stands for **A**ction, **P**rocess, **O**bject, and **S**chema. APOS Theory is a theory of how mathematical concepts can be learned.

- APOS Theory focuses on models of what might be going on in the mind of an individual when he or she is trying to learn a mathematical concept and uses these models to design instructional materials and/or to evaluate student successes and failures in dealing with mathematical problem situations.

- APOS is a constructivist theory (that says learners construct knowledge rather than just passively take in information).

Action in APOS theory

- In APOS, an **A**ction is a transformation of a mathematical Object that the individual perceives as external.
- This is in the sense that each step of the transformation must be performed explicitly, with one step serving as a prompt for the following step, and the steps cannot yet be imagined.
- An Action may be the rigid application of an explicitly available algorithm or of a memorized procedure or formula.
- An Action will typically appear as the need to do explicit computations and the individual not yet being able to discuss the notion in general terms. In particular, he/she will not be able to justify the Action.

Process in APOS theory

- When an Action is repeated, and the individual reflects on the Action or on a chain of Actions, it might be interiorized into a Process.
- A Process is perceived as internal and this allows the individual to omit steps, anticipate results, and thus generate dynamical imagery of the Process, without having to explicitly perform it.
- The individual will also be able to justify the Process. The relations the individual has established as a result of reflection will also allow to explain or discuss the Process in general terms and might allow perceiving the Process as independent of representation.
- Different Processes may be coordinated or reversed to form new Processes.

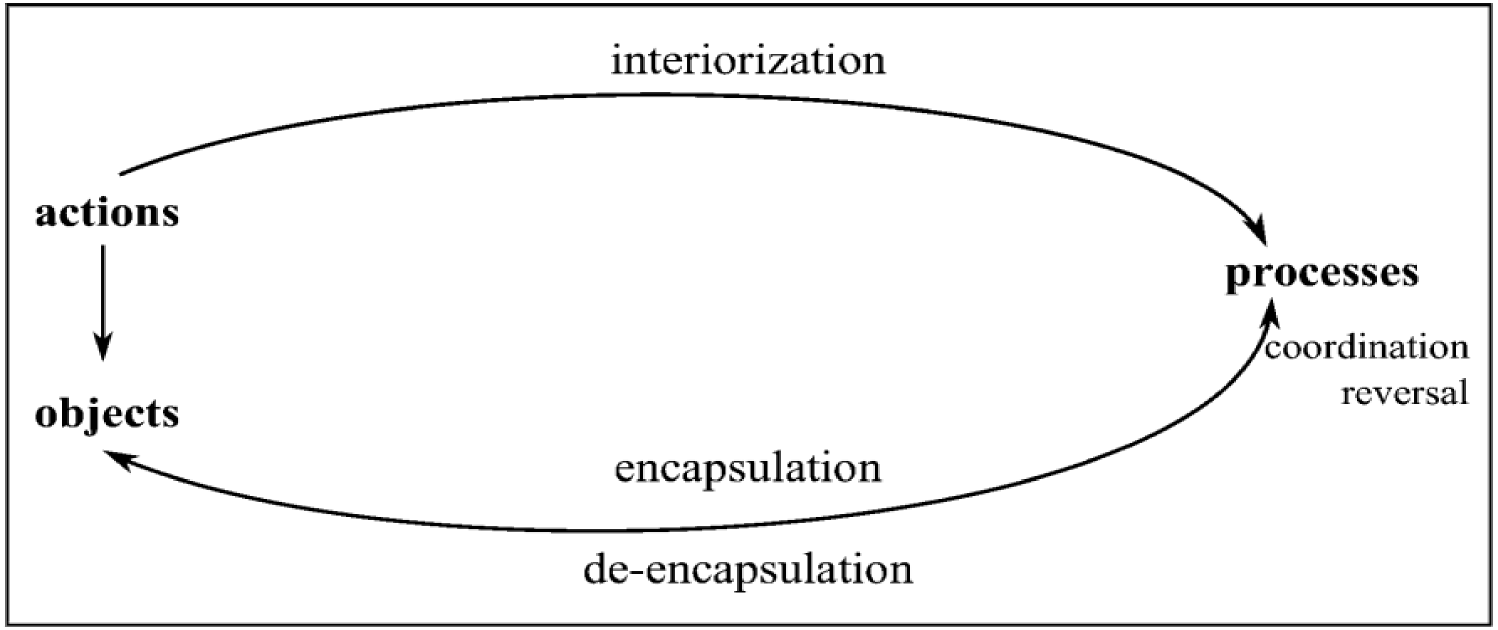
Object in APOS theory

- As the context in which a Process is constructed changes, the individual may need to solve new problem situations that require thinking of the Process as an entity in itself, then one says that the Process has been encapsulated into an Object.
- An Object may be de-encapsulated into the Process it came from, as needed in a problem situation.
- What is important about an Object is that the individual can do Actions on the (de-encapsulated) Process.

Schema

- A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas having to do with a particular mathematical notion or topic.
- The Schema is coherent in the sense that its components are related in such a way that the individual is able to decide when a problem situation fall within the scope of the Schema.

Schema



Genetic Decomposition (GD)

- Research in APOS typically starts by proposing a hypothetical model (a conjecture), in terms of the structures (Actions, Processes, Objects) and mechanisms (interiorization, coordination, reversal, encapsulation, de-encapsulation) of APOS, of how a generic student may construct a specific mathematical notion. This model is called a genetic decomposition (GD).

Some remarks for GD:

- ✓ A genetic decomposition is stated in terms of the constructs of the theory and is obtained from the researchers' mathematical knowledge, teaching experience, and any available data or previous study.

- ✓ A GD need not be unique, different researchers or even the same researcher can propose different GD's. Also, a GD is not claimed to be the best way to construct a notion and is not meant to be the definitive way to teach a particular mathematical topic.

Some remarks for GD:

- ✓ A GD needs to be tested with student interviews. What typically happens is that it may be found that students don’t do some of the proposed constructions and unexpected constructions may be inferred from students’ problem-solving activities.

- ✓ This leads to revising the GD in order to include more detail aiming to help students do the proposed constructions they seem not to be doing, to take into account unexpected constructions, and to build on what students can actually do.

- ✓ A GD serves as a preliminary hypothesis that can be successively refined as a result of experimentation.

ACE teaching cycle

The design and implementation of instruction using APOS Theory

Implementation is usually carried out using the **A**ctivities worked in small groups of students, **C**lass discussion, and **E**xercises for the home (ACE) Teaching Cycle, an instructional approach that supports development of the mental constructions called for by a genetic decomposition.

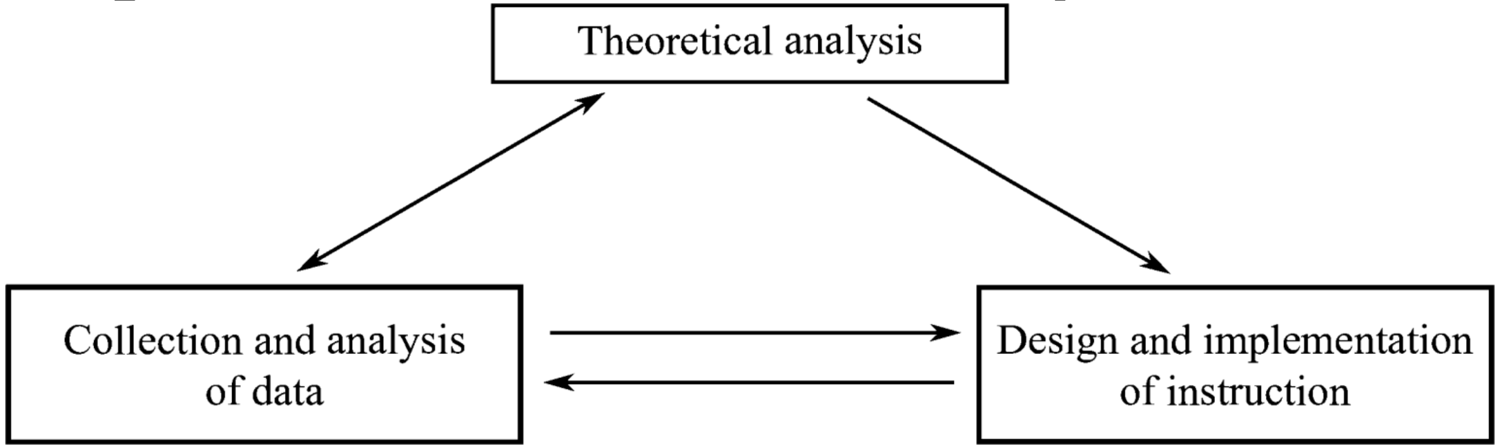
Methodology in an APOS-based research

An APOS-based research and/or curriculum development project involves three components: **a)** theoretical analysis, **b)** design and implementation of instruction, and **c)** collection and analysis of data.

a) According to this paradigm, research starts with a theoretical analysis of the cognition of the mathematical concept under consideration. This gives rise to a preliminary genetic decomposition of the concept.

b) The theoretical analysis drives the design and implementation of instruction through activities intended to foster the mental constructions called for by the analysis. Activities and exercises are designed to help students construct the mental constructions in the GD.

c) The implementation of instruction provides an opportunity for the collection and analysis of data, which is carried out using the theoretical lens of APOS Theory.



Research literature: Students' difficulties regarding trigonometric functions (sine and cosine and their inverse)

- ✓ Numerous studies have documented that students frequently show limited understanding of basic ideas of trigonometry.

- ✓ Bagni (1997) noted that more than 80% percent of the 67 Italian high school students in his study could provide a complete or partial solution to easy trigonometric equations such as, find all real values x such that $\sin x = -1/2$ or $\cos x = 1/2$, *by remembering* and *mentally reversing a memorized table* of the values of trigonometric functions of integer multiples of Pi/6, and Pi/4.

Research literature:

	$\theta = 0$ $(0°)$	$\pi/6$ $(30°)$	$\pi/4$ $(45°)$	$\pi/3$ $(60°)$	$\pi/2$ $(90°)$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—

- ✓ He reported, that more than half of the students tested produced wrong answers or no answer to questions such as find all real values x such that $\sin x = 1/3$, $\sin x = Pi/3$, or $\cos x = -\sqrt{3}/3$.

- ✓ Weber (2005) noted that while algebraic functions deal with arithmetic operations and procedures, the reasoning behind trigonometric functions arises from the geometric realm in which construction and measurement are important implicit ideas.

- ✓ Weber (2005) conjectured that students need to be able to imagine the process which gives rise to the unit circle definition of the trigonometric functions in order to be able to have an understanding of the sine and cosine functions that goes beyond the repetition of memorized facts and procedures (the understanding shown by students in Bagni’s 1997 study).

- ✓ Weber (2005) observed what the students seemed to lack was the ability or inclination to mentally or physically construct geometric objects to help them deal with trigonometric situations

Two studies (Akkoc, 2008; Topçu et al., 2006) characterized preservice and in-service mathematics teachers in Turkey as holding understandings of radian angle measures dominated by degree measure;

when given radian measures, the teachers converted these measures into a number of degrees in order to attribute a meaning to the measures. Not one of the four teachers interviewed by Topçu and colleagues defined radian measures as a ratio of lengths.

- ✓ As Akkoc (2008) reported and compatible with previous findings (Fi, 2003), preservice teachers also claimed that radian measures are only given in terms of π, leading teachers to interpret 30 as a number of degrees in expressions such as sin(30).

- ✓ In light of his findings, Akkoc suggested that impoverished radian angle measure understandings likely contribute to teacher and student difficulties in trigonometry.

I now discuss only some parts of the genetic decomposition for a unit circle based introduction to the sine and cosine, because of the limitation of time I will not explain the constructions of their inverse.

The t → P(t) process

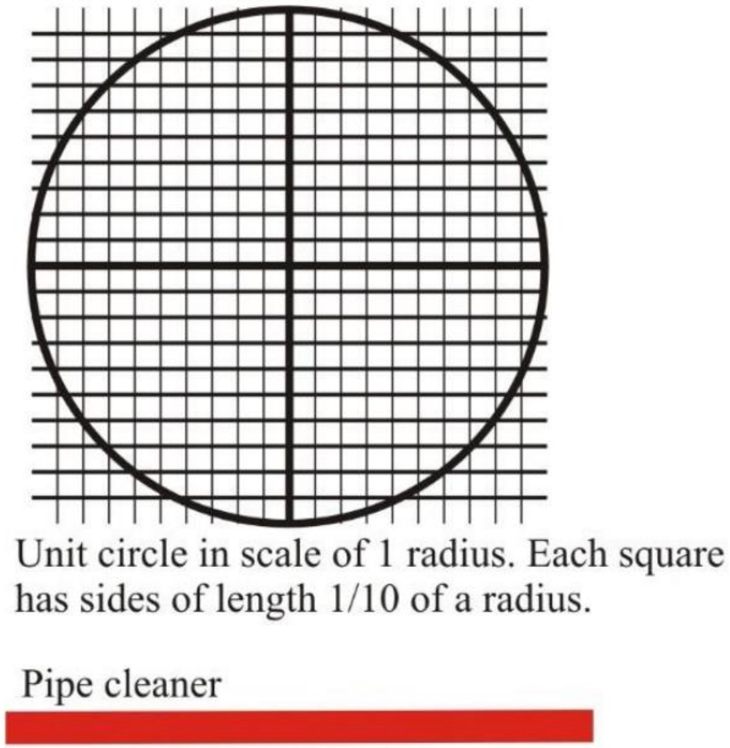
Construction of the sine and cosine functions starts with the action of taking a given real number t (preferably not an integer multiple of $\frac{\pi}{4}$ or $\frac{\pi}{6}$, at the beginning) and locating, as a geometric representation, the terminal point *P(t)* of an arc along the unit circle that starts at the point (1,0), has length $|t|$ and is traversed either counterclockwise when t is positive or clockwise in the case that t is negative.

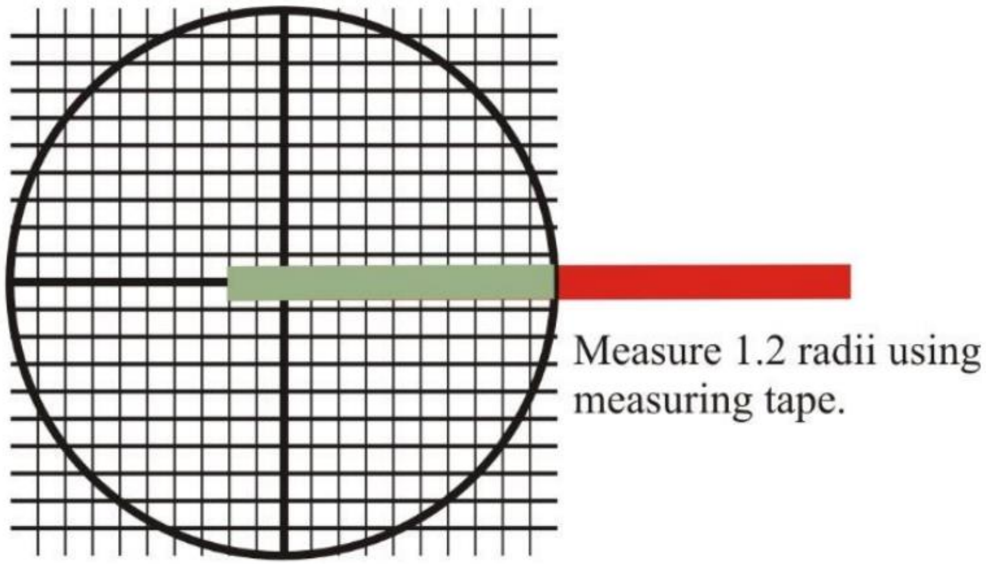
The t → P(t) process

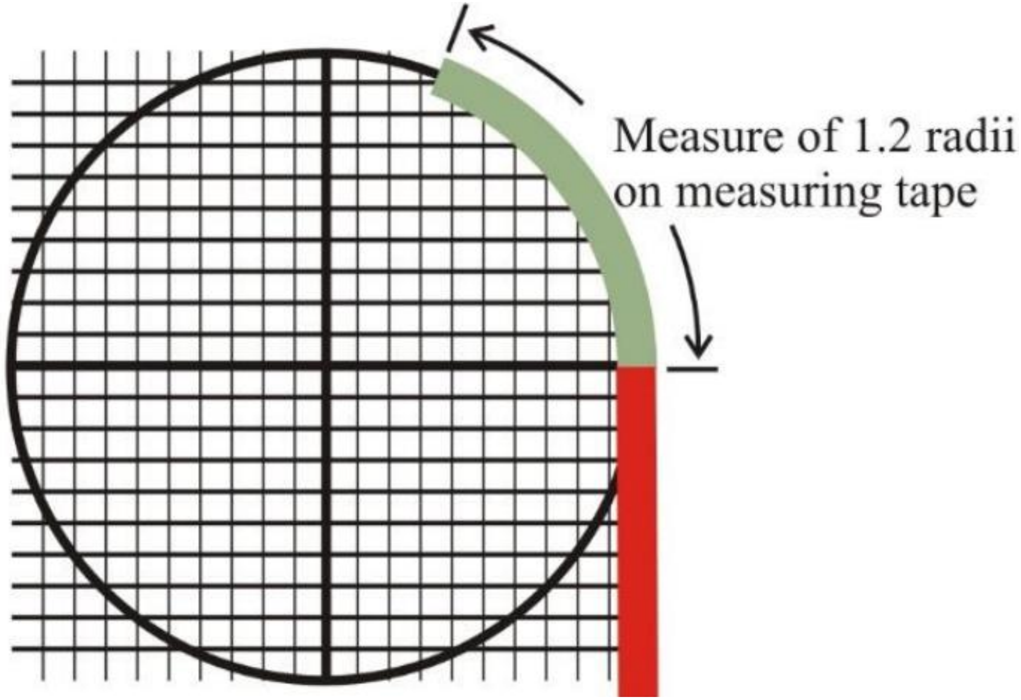
For the case of t = 1.2, where the student physically measures a length (in units of number of radii), then physically winds it around a circle placed in a Cartesian plane and locates the corresponding point in the plane. Note that at this moment P(t) is the geometric (figurative) representation of a point on the unit circle.

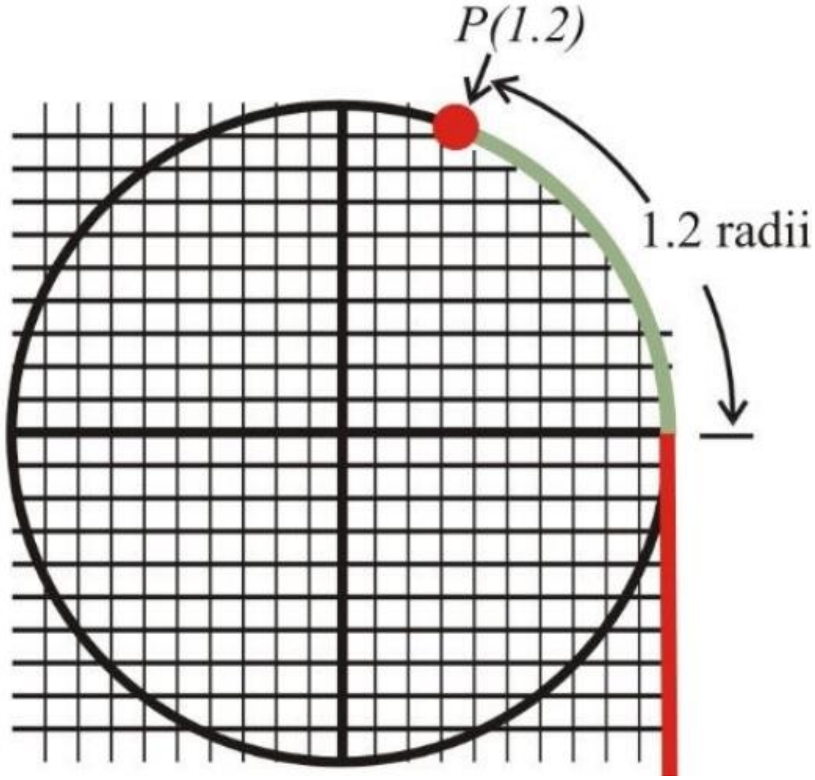
- **Point determined by a number:** Given a circle and any real number t, that number determines a point $P(t)$ on the circle.

For example given 1.2, *P(1.2)* is determined as follows:





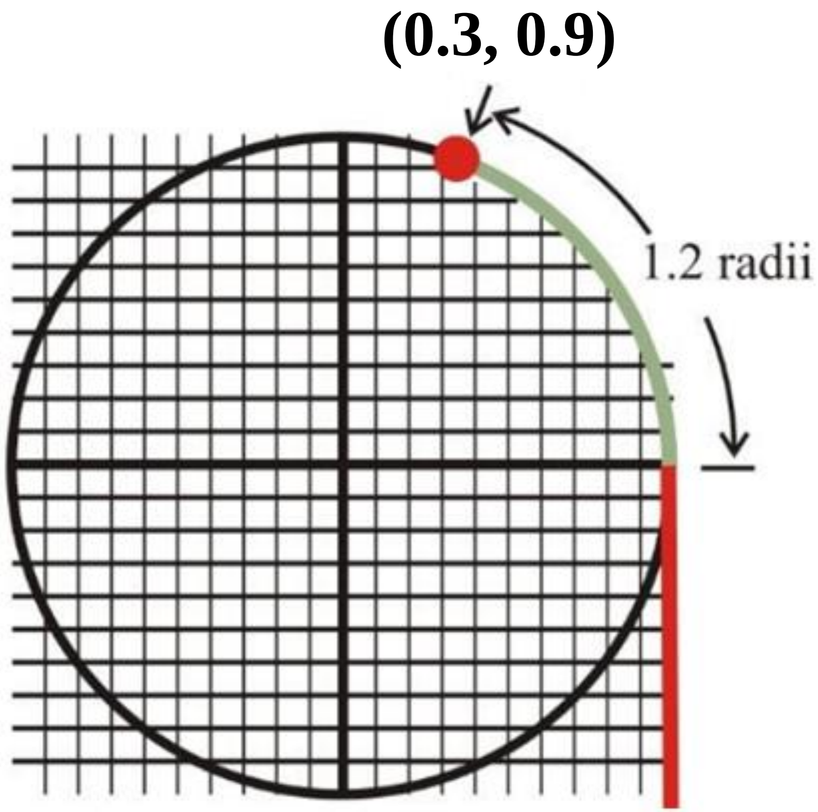

Measure of 1.2 radii
on measuring tape


P(1.2)
1.2 radii

The t → P(t) process

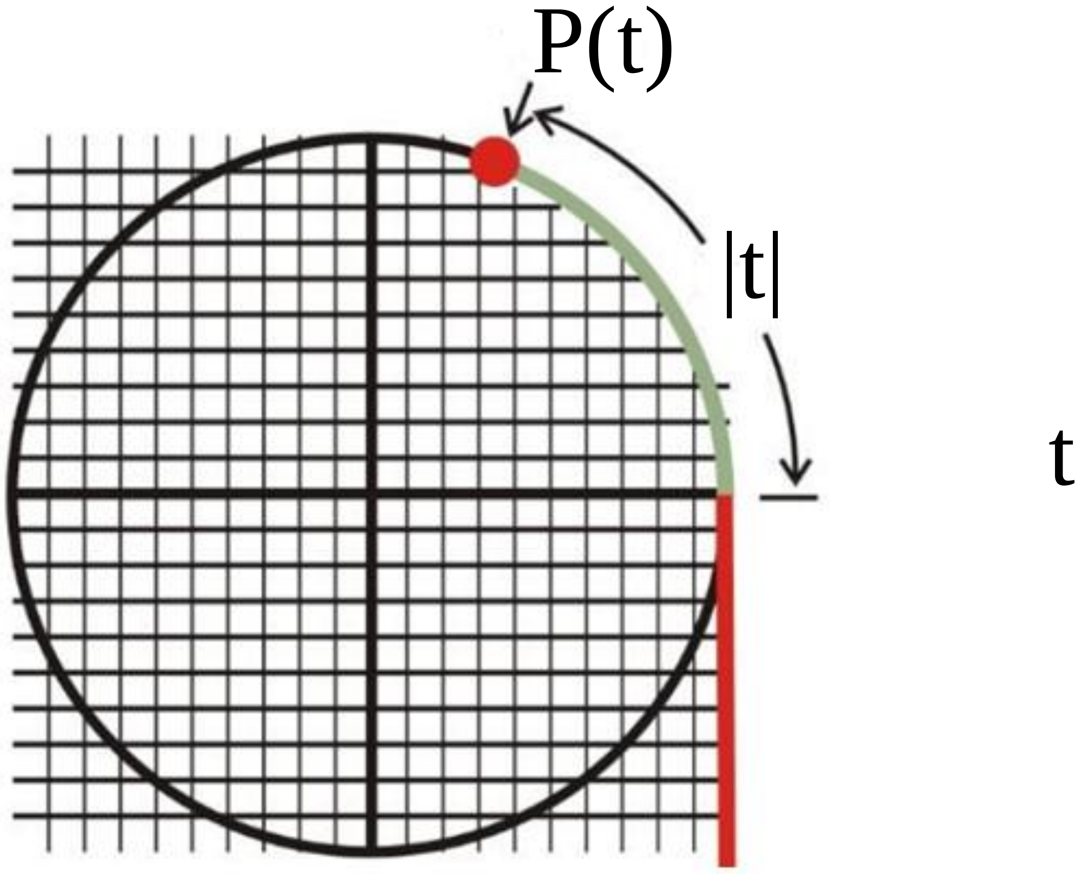
As students repeat and reflect on this action they may be able to imagine taking any given real number *t* and assigning to it a point *P(t)* on the unit circle without having to do so explicitly.

In this case they can be said to have interiorized the action into a process, denoted t → P(t). At this stage of the construction, to obtain a symbolic representation of the point P(t) as an ordered pair, students would have to physically approximate the coordinates of the point.

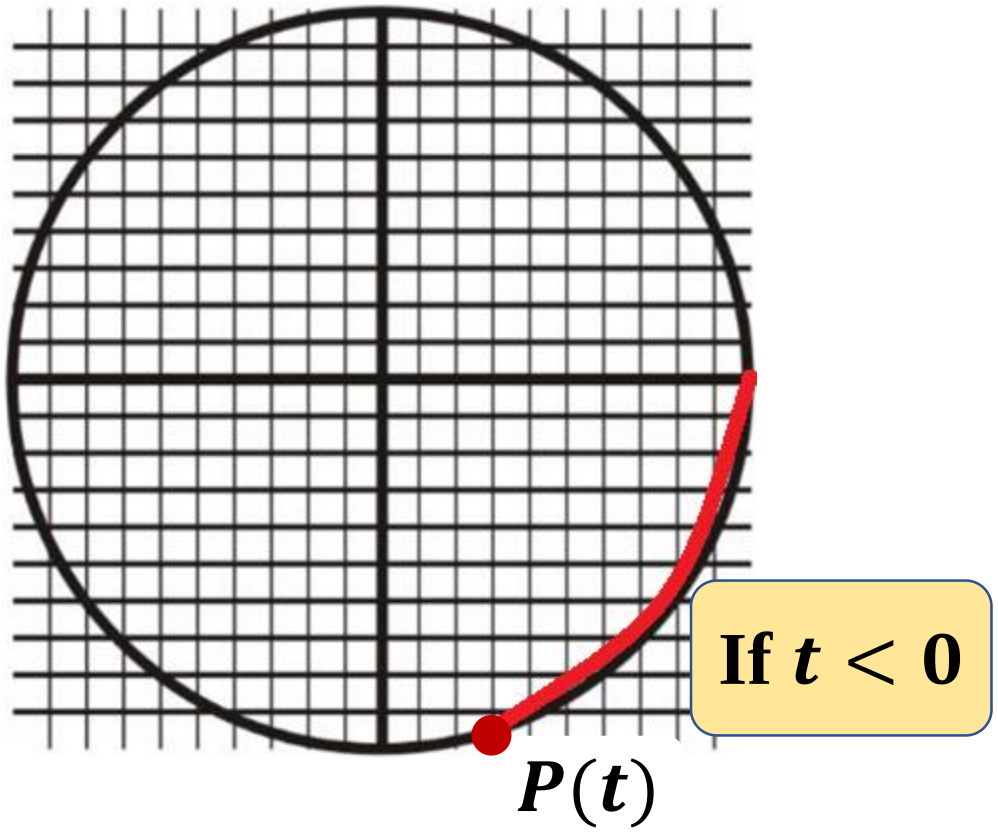


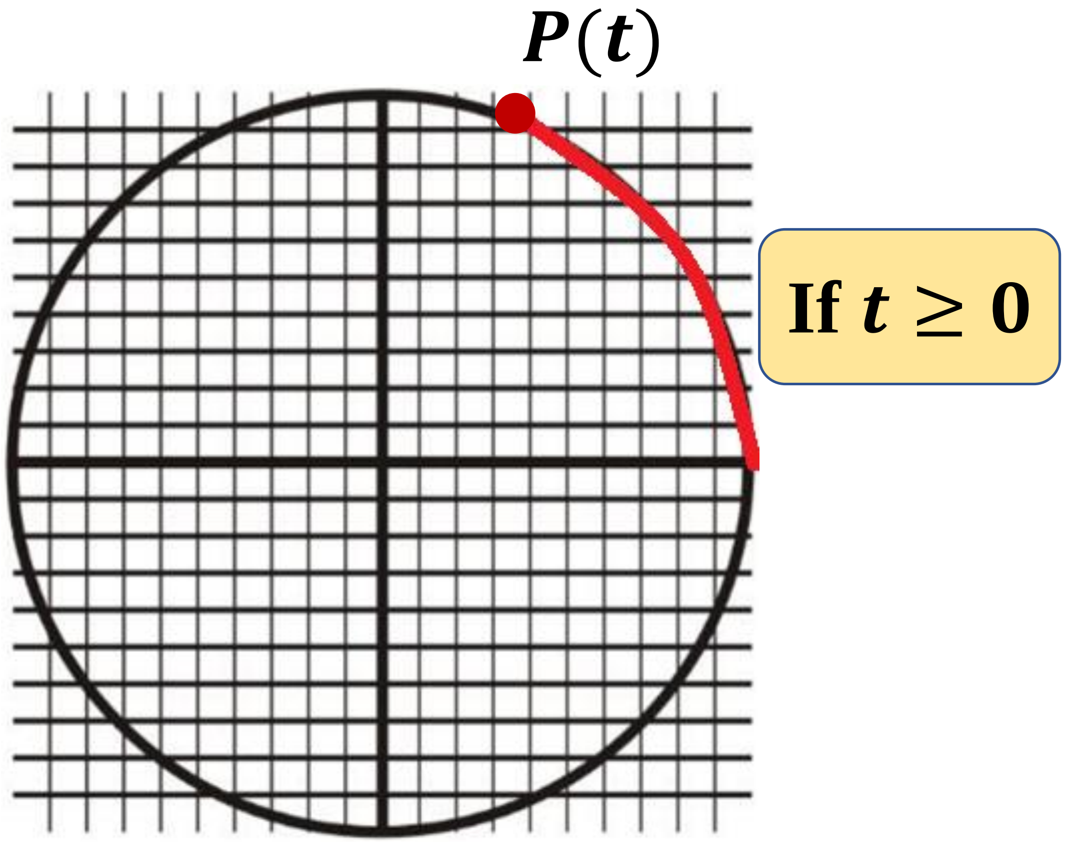
The $t \rightarrow P(t)$ process

The process $t \rightarrow P(t)$ may be described as a change of representation that starts with a symbolic representation of a number t, converts it to a directed circular arc having length of $|t|$ radii, and ends with a geometrical representation of the end point $P(t)$ of the arc. As such, as is commonly the case with conversions between representations, this can be the cause of student difficulty and needs to be given explicit attention (Duval, 2006).



- If $t \geq 0$ the arc is measured counter-clockwise and if $t < 0$ the arc is measured clockwise. The point where you end is *P(t)*.





The projection processes

Now a process conception of the sine and cosine functions may be constructed by coordinating the $t \rightarrow P(t)$ process with a corresponding projection process.

- Projecting onto the y axis defines the sine function,
- and projecting onto the x axis defines the cosine function.

The projection processes

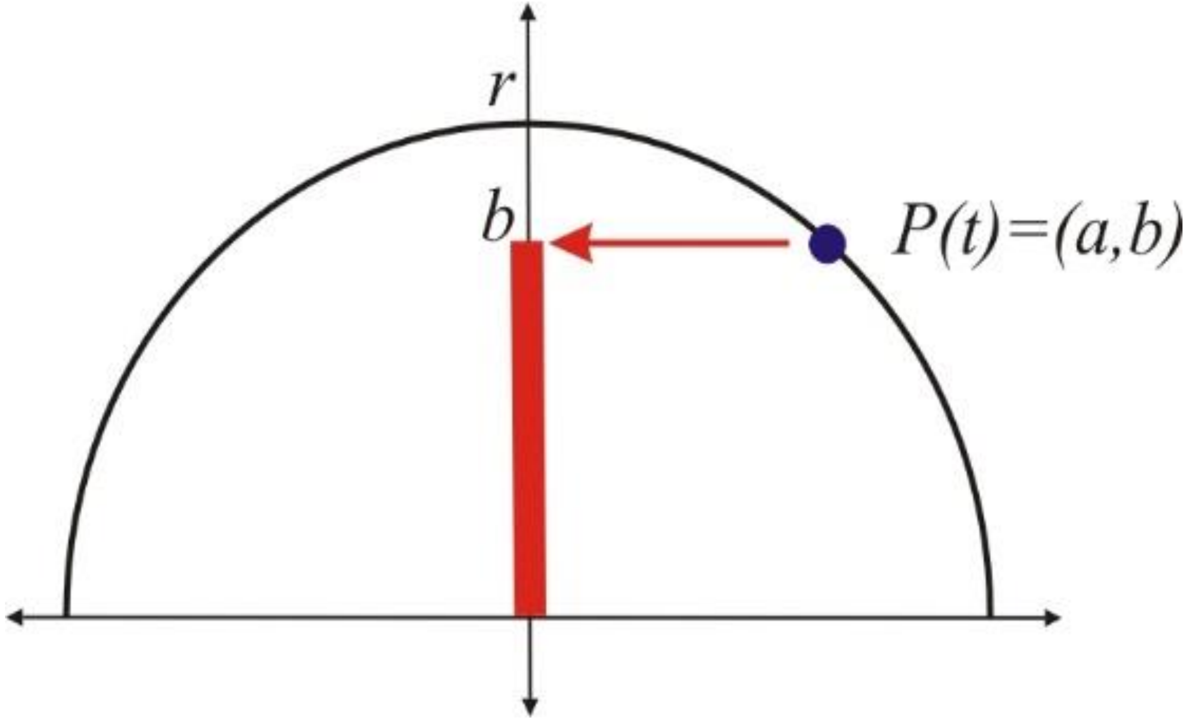
By “projection” – in case of the sine function – we mean performing the action of taking the point P(t) as input and producing a geometric representation of the point on the y axis that is at the same height (that is, that has the same y coordinate), and following this with the action that takes the point so obtained on the y axis (its geometric representation) as an input and produces its y coordinate (a number).

This “projection” is not trivial, as it involves relating different representations of a mathematical object and hence requires explicit attention during instructional activities (Duval, 2006). These actions of projection may be interiorized into processes of “projection”. The processes of locating a corresponding point P(t) and then projecting onto a corresponding axis (as described above) may be coordinated into processes which we will refer to as the definition of the sine and cosine functions.

Given any real number *t*, the **sine of *t*** is defined, $\mathbf{sin}(\boldsymbol{t})$, as follows:

- First you locate *P(t)* in a circle
- Then you project *P(t)* on the *y-axis* and give the result in units of radii

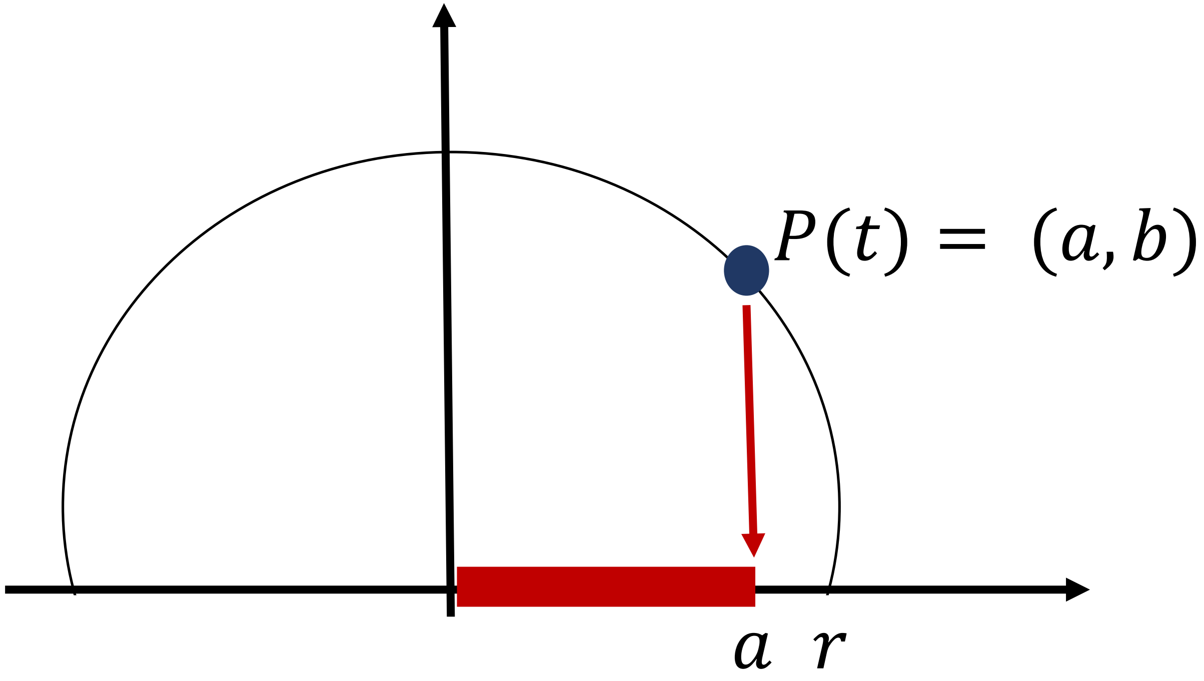
$$\boldsymbol{sin}\,(\boldsymbol{t}) = \frac{\boldsymbol{b}}{\boldsymbol{r}}$$



Given any real number t, the **cosine of t** is defined, $\mathbf{cos}(\boldsymbol{t})$, as follows:

- First you locate *P(t)* in a circle
- Then you project *P(t)* on the *x-axis* and give the result in units of radii

$$\boldsymbol{cos\,(t) = \frac{a}{r}}$$



The projection processes

Our genetic decomposition emphasizes that the definition of the sine and cosine functions requires two separate steps:

1) The first step is to assign a point on the circle to the given real number.
2) The second step is to project that point onto the appropriate axis.

- It must be pointed out, as has been observed by other researchers, for example Bagni (1997) and Weber (2005), that the above constructions need also be done for real numbers that are not integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Note that the above constructions may be interiorized into the recognition that the set of real numbers is the domain of the sine and cosine functions and that the range of these functions is the interval $[-1, 1]$.

(In class) You should be able to do this exercise without using a calculator, thread, or wikistick. Use only your imagination (imagine locating and then projecting).

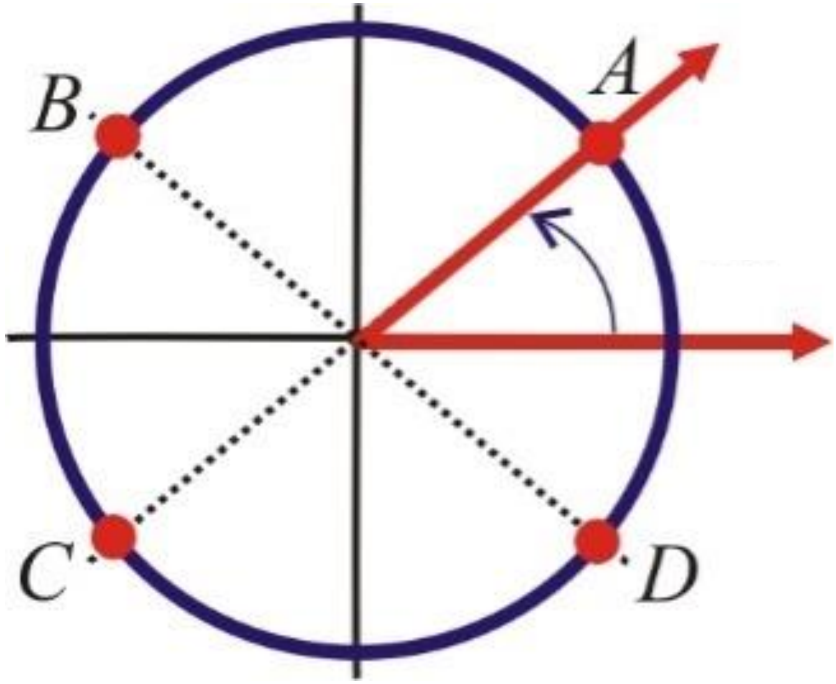
Put in order from lowest to highest:

$$\cos(1)\,,\ \cos\left(\frac{1}{10}\right),\ \cos(3)\,,\ \cos(-4)\,,\ \cos(4)\,,\ \cos(6)$$

The Circ process

In another construction, given a point P(t), represented geometrically or as an ordered pair, the student must perform the action of finding the other three corresponding points P(−t), P(t +), and P(− t) on the unit circle, in the respective representation, geometrically or as ordered pairs (see Fig. 3). These actions are interiorized into a process that enables students to locate on the same circle the geometric representation of any point of the form P(t + n2), P(−t + n2), P(t +   + 2n),P(− t + 2n) when they know a geometric representation for the point P(t). The process would also enable students to find the coordinates of any of those points given those of P(t). Observe that the interiorization of these actions into a process should lead to the recognition that the mapping P(t) is one to one only when its domain is restricted to intervals of length 2, to the periodicity of the map, and to an awareness that an infinite number of t are assigned to the same point P(t). The process as described above will be denoted Circ and will be referred to as the process of symmetries of the circle (only symmetries generated by reflections across the axes and origin are considered). That is, Circ refers to the set of transformations described above where given the geometric representation of a point P(t) on the unit circle with one or both of its coordinates and a number of the form t + n or −t + n (n, an integer), the student reflects and/or rotates the point P(t) on the circle to obtain a new point P(t + n) or P(−t + n) (first in its geometric representation), and follows this with the process of obtaining the coordinates of the new point using those of P(t).

A point on a circle determines three other points on the circle, reflecting through each axis and through the origin.



Inverse trigonometric functions

To define the inverse trigonometric functions, our genetic decomposition will now reverse the process of definition of the sine and cosine trigonometric functions. Not much can be found in the mathematics education research literature regarding the inverse trigonometric functions. We do not claim that our genetic decomposition is the only way to organize and/or describe the mental constructions students may do in order to understand inverse trigonometric functions. Neither do we claim that this is the way students actually think of inverse trigonometric functions. What we claim is that interiorization into processes of the actions to be described can help students build an understanding of inverse trigonometric functions.

Reversal of the projection

So we now proceed to describe what we will call the reversal of the definition of the sine and cosine functions. To reverse the projection of the sine function, start with a number k in the interval $[-1, 1]$ and perform the action of representing a point on the y axis that has k as its ordinate. The next action is to locate on a physical or geometric representation of the unit circle all the points that are projected horizontally onto (0,k). There is one point if $k = \pm 1$ and two corresponding points if $-1 < k < 1$. To reverse the projection of the cosine function, perform the analogous action, namely represent a point on the x axis having abscissa k, and then identify all points on the unit circle that project vertically onto (k,0). Repetition and reflection on these actions may be interiorized into a process of projection reversal. We stress that the projection reversal starts with numbers as inputs and produces the geometric representation of points on the unit circle as outputs. The symbolic representation of those points as ordered pairs of real numbers could be obtained coordinating this process of reversal of a projection with process conceptions of the equation of the unit circle, or, equivalently, the Pythagorean Theorem. The process that starts with a real number k and ends with all points on the unit circle that project horizontally (in the case of the sine function) onto (0,k) is called the reversal of the projection. Similarly for the cosine function.

Reversal of the t → P(t) process

After reversing the projection, the t → P(t) process needs to be reversed. To do this, the student starts with one or two points P(t) resulting from a projection reversal and finds a value of t determining one of the points. At this stage, finding an approximation of a real number t that determines a point P(t) with a specific x or y coordinate may be done physically with a manipulative as the one represented in Fig. 2. This may also be done with a geometric argument in the case of special values. The process that starts with the output of the projection (one or two points P(t) on the unit circle having the same y coordinate- in the case of the sine function) and ends with a value of t that determines at least one of the points is called the reversal of the t → P(t) process.

Reversal of the definition

After a student reverses a projection and obtains the points that correspond on the unit circle, the student may coordinate the reversal of the $t \rightarrow P(t)$ process (to obtain one value of t) with the Circ process to obtain all values of t that determine the points he/she found on the unit circle. The reversal of the $t \rightarrow P(t)$ process followed by the coordination with Circ results in a process that allows the student to recognize that the sine and cosine functions are not one to one. The chain of actions that starts with a number, represents it as an x or y coordinate on the corresponding axis, goes on to identify the point or points on the unit circle having that number as an x, or respectively y coordinate, and then identifies all the real numbers corresponding to the point or points on the unit circle, may be interiorized into a process that we will call reversal of the definition. This process starts with a coordinate and produces the collection of all real numbers corresponding to the points on the unit circle (one, two, or none) having that coordinate. By its nature this process does not define a function.

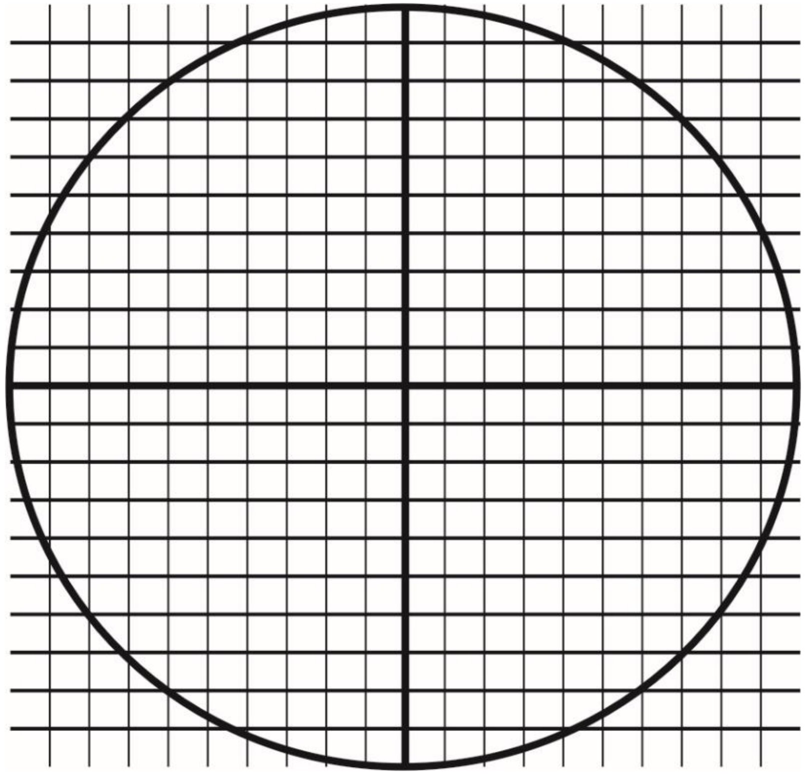
The Range process

The coordination of a Range process with a process conception of function is needed in the mental construction of a process conception of inverse trigonometric functions. In particular, the coordination of the process of reversing the definition with, what we will call, the process of Range (of the corresponding inverse trigonometric function), is interiorized into a process of inverse trigonometric function. To construct a process conception of Range, students could interiorize actions that explore ways of restricting the domain of the sine and cosine functions to an interval so that the resulting function is one to one and the restricted domain is as large as possible. These actions should include both, the unit circle representation and the graphs of these functions. Students that interiorize these actions into a process would recognize the need to restrict the domains of sine and cosine as well as the convenience of restricting these domains as they normally are. Students not able to argue for the need of a restriction and reasonableness of the usual restrictions of the domains of sine and cosine will be constrained to having an action conception of Range as a memorized fact. The construction of a Range process can be expected to be a source of student difficulty as it requires the coordination of several different process conceptions. Hence, to summarize, a process of inverse trigonometric function can result from the interiorization of the process of reversing the definition followed by coordination with a process of Range. Once students have a process of inverse trigonometric functions, it may be coordinated with processes that use technology to explore the mental constructions that have been made. To summarize, the genetic decomposition underscores that the construction of a process conception for the sine and cosine functions and their corresponding inverse functions depends on the construction of the mental processes $t \to P(t)$, the reversal of $t \to P(t)$, the projections, the reversal of the projections, Circ, and Range. These processes will be referred to as “the basic processes” or “basic mental constructions”.

Inverse sine

Suppose we have the value of u between -1 and 1, in $\sin t = u$ the unique answer for t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is called $\sin^{-1}(u)$ ("inverse sine of u").

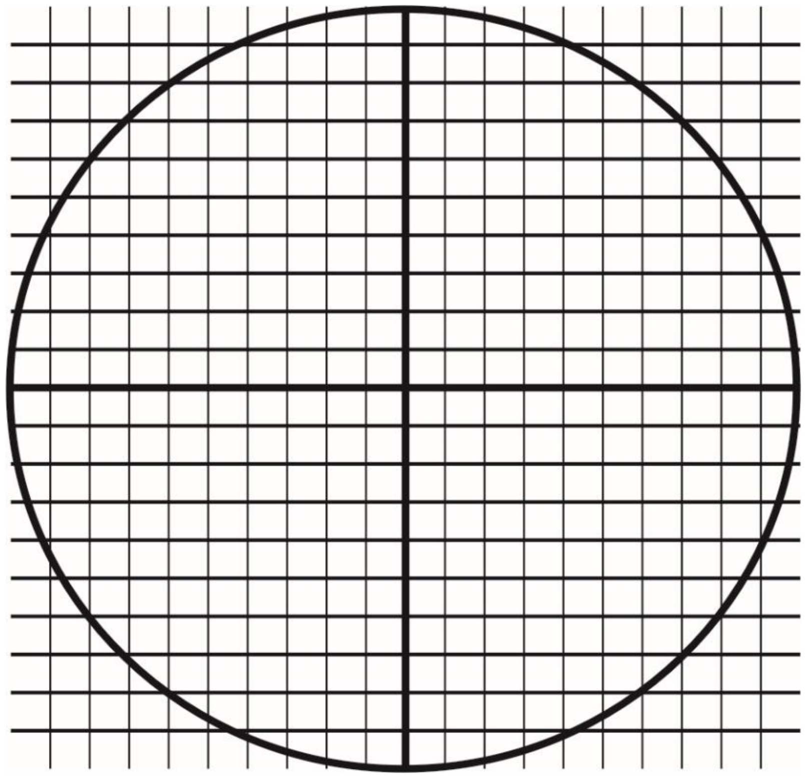
$$t = sin^{-1}(u)$$



Inverse cosine

Suppose we have the value of u between -1 and 1, in $\cos t = u$ the unique answer for t between 0 and π is called $\cos^{-1}(u)$ ("inverse cosine of u").

$$t = cos^{-1}(u)$$



The set of activities to help students make the mental constructions in the GD

Thank you so much for your attention